Hölder properties of Weierstrass-like solutions of θ-twisted cohomological equations

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Abstract

It is proved that bounded solutions of modified (θ-twisted) cohomological equations for expanding circle maps are θ-Hölder continuous but are not $(\theta+\gamma)$-Hölder continuous for every $\gamma > 0$ at almost every point. This gives new examples of "nonlinear" Weierstrass-like functions for which the optimal Hölder exponent at most points is known.

Contents

1 Introduction

In 1895 Weierstrass [24] constructed an example of a continuous nowhere differentiable function

$$W(x) = \sum_{n=0}^{\infty} a^n \cos(2\pi b^n x), \quad x \in \mathbb{R},$$

where $0 < 1/b < a < 1$. Hardy [13] for $\theta = -\log a/\log b$ proved that at any point x this function is θ-Hölder but it is not $(\theta+\gamma)$-Hölder for any $\gamma > 0$.

Hardy's result has been generalized [6] to the case when cos is replaced by any non-constant Lipschitz function v from an open dense subset of the space of 1-periodic Lipschitz functions $g : \mathbb{R} \to \mathbb{R}$ (with its standard norm).

Some other properties and examples of Weierstrass-like functions were considered in [8, 14, 10, 16, 15, 6, 7].

There was a parallel research in the fractal dimension of graphs of functions like $W(x)$ (see [17, 20, 22, 2, 18, 23]). It is known that the bounds on a fractal dimension (either box-counting or Hausdorff) of the graph of a function give bounds on the global Hölder exponent of the function (see [11, 20, 8], for example).

It is important that all the research mentioned above applies only to the linear case when the multipliers in front of cos (or its replacement) and the one in the argument of cos are exactly n-th powers of some numbers.

For $0 < \theta \le 1$, C^2-smooth $f : S^1 \to S^1$ with $f'(x) > \lambda > 1$, $\forall x \in S^1$ and $v \in C^{1+\varepsilon}(S^1)$ consider

$$\alpha(x) = -\sum_{i=0}^{\infty} \frac{v(f^i(x))}{((f^{i+1})'(x))^{\theta}}. \tag{1.1}$$

Functions of this kind are similar to the Weierstrass one (if one considers S^1 as $\mathbb{R}/\mathbb{Z}$) and for $\theta = 1$ they correspond to solutions of twisted cohomological equations (2.1) which are important in the study of linear response for one-dimensional chaotic dynamical systems (see [1]).

For $\theta = 1$ the modulus of continuity of α is thoroughly studied in [9]. When f is an Anosov diffeomorphism, Hölder properties of α are studied in [12].

Note that Weierstrass-like functions are related to certain two-dimensional discrete dynamical systems. In particular to compute fractal dimensions of

the graph of W it is useful to interpret it as a repellor of following dynamical system (see [2], for example):

$$G : (\mathbb{R}/\mathbb{Z}) \times \mathbb{R} \to (\mathbb{R}/\mathbb{Z}) \times \mathbb{R},$$
$$G(x, y) = \left(bx(\mod 1), \frac{y - v(x)}{a} \right).$$

Papers [4, 3, 21] use this approach and give expressions for fractal dimension(s) of the graph of α for the case of one-dimensional (piecewise)-expanding maps. However these expressions involve quantities from thermodynamical formalism that are difficult to compute explicitly in general. In fact even if dimension can be computed explicitly, if one is interested in knowing Hölder exponents at most points (rather than knowing only a global exponent), information about fractal dimension of the graph of the function is not enough.

In this paper the study of Hölder continuity properties of α for the case $0 < \theta < 1$ is presented. This gives new examples of dynamically-defined functions that have at almost every point a globally prescribed Hölder exponent that can not be improved.

The paper demonstrates two ways of studying Hölder properties of functions like α. First uses the repellor representation of the graph of α and coding of the dynamics of f by a shift on a symbolic space. The second one is rather elementary, it is based on bounded distortion and density of most trajectories.

Interestingly, an upper bound for the optimal Hölder exponent is much simpler to obtain by following the first way and a lower bound – by following the second way.

2 General definitions and main results

Let f be a C^2-smooth endomorphism of S^1 such that $\lambda = \min_{y \in S^1} f'(y) > 1$. Assume $0 < \varepsilon \leq 1$ and $0 < \theta < 1$. Let $v : S^1 \to \mathbb{R}$ be a $C^{1+\varepsilon}$ function.

For a positive natural number r define $E_r(x) = rx(\mod 1)$.

The following Lemma will be used to state the main result and in proofs:

Lemma 2.1. *For every natural number $r \geq 1$ there exists only one bounded solution $\alpha : S^1 \to \mathbb{R}$ to the θ-twisted cohomological equation*

$$v(E_r(x)) = \alpha(f(x)) - (f'(x))^\theta \alpha(x). \tag{2.1}$$

This solution is given by the following formula:

$$\alpha(x) = -\sum_{i=0}^{\infty} \frac{v(E_r(f^i(x)))}{((f^{i+1})'(x))^\theta}, \quad x \in S^1. \tag{2.2}$$

Remark 2.2. Note that in the case $r = 1$ the formula above gives exactly the function (1.1) mentioned in Introduction.

Definition 1. For $\omega : I \to \mathbb{R}$ its (optimal) Hölder exponent at point x_0 is the following quantity

$$h_x(\omega) = \varliminf_{\varepsilon \to 0} \left\{ \frac{\log |\omega(x) - \omega(y)|}{\log |x - y|} \mid y \in B_\varepsilon(x) \right\}.$$

Definition 2. For $\omega : I \to \mathbb{R}$ its (optimal) global Hölder exponent is just $h(\omega) = \inf_{x \in I} h_x(\omega)$.

Theorem 1. *Let α be the only bounded solution to equation* (2.1) *for $r = 1$. There are only two possibilities:*

1. *α is a C^1-smooth function*
2. *$h_x(\alpha) = \theta$ for Lebesgue almost-every x.*

Remark 2.3. In fact in possibility 2 the inequality $h_x(\alpha) \geq \theta$ holds everywhere. See Theorem 3.

Remark 2.4. Note that the formula (2.2) for the solution to equation (2.1) resembles the formula for the classical Weierstrass function (1).

Put $\theta = -\log a / \log b$. The classical result of of Hardy states that the Weierstrass function is θ-Hölder at every $x \in \mathbb{R}$ but for $\gamma > 0$ is not $(\theta + \gamma)$-Hölder at any $x \in \mathbb{R}$. If one puts $f(x) = E_b(x)$ and $v(x) = \cos(2\pi x)$, then Theorem 1 implies an "almost-every" version of Hardy's result.

We will split the proof of the Theorem 1 into two parts.

First in the Section 3.2 we prove a version of Theorem 1 where we have $\leq$ sign instead of $=$ in the possibility 2.

Next in the Section 4 we prove that $h_x(\alpha) \geq \theta$ always and for every x, thus finishing the proof of 1. In that section we also present a different proof of a weaker version of Theorem 1 without referring to symbolic dynamics.

3 Study of Hölder exponents using symbolic dynamics

In this section we will prove most of the Theorem 1 following ideas from [5]. First we introduce some notations and discuss a related result from [4].

3.1 Box-counting dimension of the graph of α

Denote $I = [0, 1]$. We identify S^1 and $I/\sim$ where $\sim$ identifies 0 and 1. Slightly abusing notation we identify f and its lift with respect to this factorization by $\sim$.

Let $Y = I \times \mathbb{R}$ and a C^1 map $G : Y \to Y$ of the form $G(x,y) = (f(x), \bar{f}(x,y))$. Suppose $\bar{f}(x,\cdot)$ is an expanding diffeomorphism of $\mathbb{R}$ and

$$\left|\frac{\partial \bar{f}(x,y)}{\partial y}\right| < \left|f'(x)\right|. \tag{3.1}$$

Denote by n_0 the topological degree of f. Denote by $\{l_i\}_{i=0}^{n_0-1}$ the sequence of points dividing into segments of bijectivity of f such that $l_0 = 0$ and $l_{n_0-1} = 1$. As f is n_0-to-1, in particular we have that Image $f|_{[l_i,l_{i+1}]} = I$. Put $\phi_i = (G|_{[l_i,l_{i+1}]})^{-1}$ for $0 \le i < n_0$ and assume that ϕ_i is $C^{1+\varepsilon}$ for every i.

Obviously $\phi_i : Y \to Y$ and injective. There exist functions $f_i^{-1} : I \to I_i$ (inverse branches) and $\bar{\psi} : Y \to \mathbb{R}$ such that ϕ_i can be written in the following form: $\phi_i = (f_i^{-1}(x), \bar{\psi}_i(x,y))$, where

$$\left|(f_i^{-1})'\right| \in (0,1), \qquad \left|\frac{\partial \bar{\psi}_i(x,y)}{\partial y}\right| \in (0,1).$$

Let $a_i, b_i, c_i : Y \to \mathbb{R}$ be such that for $z \in Y$ we have

$$D\phi_i(z) = \begin{pmatrix} a_i(z) & 0 \\ b_i(z) & c_i(z) \end{pmatrix}.$$

Assumption (3.1) implies that $c_i(z) > a_i(z)$ for every z.

Consider the global repeller for G:

$$E = \{(x,y) \mid \{G^n(x,y)\}_{n=0}^{\infty} \text{ is bounded}\}.$$

Let $\Sigma = \{0, 1, \ldots, n_0 - 1\}^{\mathbb{Z}_{\ge 0}}$ be a full one-sided shift on n_0 symbols. It is well known that one can code every point of the I by a sequence from Σ. Define $\tilde{\pi} : \Sigma \to I$ and $\pi : \Sigma \to E$ by

$$\pi(x) = \bigcap_{n\ge 0} \phi_{x_0} \circ \ldots \circ \phi_{x_n}(E),$$
$$\tilde{\pi}(x) = \bigcap_{n\ge 0} f_{x_0}^{-1} \circ \ldots \circ f_{x_n}^{-1}(I).$$

As both ϕ_i and f_i^{-1} are strict contractions, intersections in the definitions of π and $\tilde{\pi}$ consist of single points.

Let σ be a left shift on Σ. It is folklore that there exist an ergodic invariant measure μ on I that is absolutely continuous with respect to the Lebesgue measure on I. Denote its push-back under the action of $\tilde{\pi}$ by μ_Σ. It is known that $\tilde{\pi}$ and π are μ_Σ-a.e. one-to-one and conjugate the dynamics of f on I and G on E with the dynamics of σ on Σ and μ_Σ is a σ-invariant measure on Σ.

The repellor E admits the following characterization:

Lemma 3.1 ([4]). *E is a graph of continuous function $\alpha : I \to \mathbb{R}$.*

For $n \geq 0$ and $x'_0, \dots, x'_n \in \{0, \dots, n_0 - 1\}$ denote the corresponding $n+1$ cylinder by

$$C_{x'_0 \cdots x'_n} = \left\{ x \in \Sigma \;\middle|\; x_i = x'_i,\ 0 \leq i \leq n \right\}.$$

For $\beta : \Sigma \to \mathbb{R}$ denote

$$S_n \beta(x) = \sum_{i=0}^{n-1} \beta(\sigma^i(x)).$$

For a continuous $\beta : \Sigma \to \mathbb{R}$ denote the topological pressure of β by $P(\beta)$. I.e.

$$P(\beta) = \lim_{n \to \infty} \frac{1}{n} \log \left(\sum_{C_n} \inf_{x \in C_n} \exp(S_n \beta(x)) \right),$$

where the summation is taken over all n-cylinders C_n of Σ.

Define two functions $f_W, f_H : \Sigma \to \mathbb{R}$ by

$$f_W(x) = -\log a_{x_0}(\pi \sigma(x)), \quad f_H(x) = -\log c_{x_0}(\pi \sigma(x)).$$

Denote the fixed points of ϕ_0 and ϕ_{n_0} by $z_0 = (0, y_0)$ and $z_{n_0 - 1} = (1, y_{n_0 - 1})$. Denote the (global) strong stable manifold of ϕ_i at point z_i by

$$W^{ss}_{\phi_i}(z_i) = Y \cap \bigcup_{n \geq 0} \phi_i^{-n}(W^{ss}_{\phi_i, loc}(z_i)),$$

where

$$W^{ss}_{\phi_i, loc}(z_i) = \left\{ z \in Y \;\mid\; \{\log |\phi_i^n(z) - z_i| - n \log a_i(z_i)\}_{n=0}^{\infty} \text{ is bounded} \right\}.$$

Is not difficult to see that $W^{ss}_{\phi_i}(z_i)$ is a graph of some function from I to $\mathbb{R}$ having a continuous derivative (see [4], p. 58 for this remark).

For completeness we give a definition of the box-counting dimension although we will not use its precise form:

Definition 3. For $\gamma > 0$ and $A \subset Y$ let $M(\gamma, A)$ be the minimum number of boxes of side length γ that are required to cover A. The box dimension (or capacity) of the set A is the following quantity

$$\dim_B(A) = \overline{\lim_{\gamma \to 0}} \frac{\log M(\gamma, A)}{-\log \gamma}.$$

The following theorem is valid:

Theorem 2 ([4]). *There are only two possibilities:*

1. stable manifolds of all z_i coincide, i.e.

$$W^{ss}_{\phi_0}(z_0) = \cdots = W^{ss}_{\phi_{n_0-1}}(z_{n_0-1}).$$

Then $E = W^{ss}_{\phi_0}(z_0)$. In particular E is a C^1-smooth manifold.

2. otherwise the box-counting dimension of E is equal to $t+1$, where $t \in \mathbb{R}$ is such that $P(-tf_W - f_H) = 0$.

Recall that v is a function from I to $\mathbb{R}$ that is $C^{1+\varepsilon}$ and choose a very special G:

$$G(x,y) = \Big(f(x), y(f'(x))^\theta + v(x) \Big). \tag{3.2}$$

Then the following lemma shows how the repeller of G is related ot the Weierstrass-like function α mentioned above:

Lemma 3.2. *The repellor E coincides with the graph of the only bounded solutions to the θ-twisted cohomological equation (Equation (3.3)).*

Proof. It is enough to solve for $\alpha(x)$ the following equation

$$G(x, \alpha(x)) = (f(x), \alpha(f(x))). \tag{3.3}$$

Its solution exists if and only if f satisfies equation (2.1) for $r = 1$. Therefore it has unique bounded continuous solution by Lemma 2.1.

As α satisfies Equation (3.3), its graph is invariant. Its boundedness implies that it is a subset of E. But E is itself a graph of a function due to Lemma 3.1, in particular for any $x \in I$ there is only one point from E with x as a first coordinate. Therefore the graph of α is equal to E. ☐

Our choice of G implies that for $(x,y) \in Y$

$$\phi_i(x,y) = (f_i^{-1}(x), y(f'(f_i^{-1}(x)))^{-\theta} + v(f_i^{-1}(x))).$$

Therefore

$$a_i(x,y) = (f'(f_i^{-1}(x)))^{-1},$$
$$b_i(x,y) = \frac{\partial \left((f'(f_i^{-1}(x)))^{-\theta} \left(y - v(f_i^{-1}(x)) \right) \right)}{\partial x},$$
$$c_i(x,y) = (f'(f_i^{-1}(x)))^{-\theta}$$

and

$$f_W(x) = \log f'(f_{x_0}^{-1}(\tilde{\pi}(\sigma(x)))), \quad f_H(x) = \theta \log f'(f_{x_0}^{-1}(\tilde{\pi}(\sigma(x)))).$$

It is very important that in our case (when G is of form (3.2)) f_H is proportional to f_W.

Recall that the map $\tilde{\pi}$ conjugates the dynamics of the shift and the dynamics of f and is bijective almost everywhere. Also our definition of $\tilde{\pi}$ implies that $\tilde{\pi}(x) \in f_{x_0}^{-1}(I)$. Thus for μ_Σ-a.e. x we have $f_{x_0}^{-1}(\tilde{\pi}(\sigma(x))) = f_{x_0}^{-1}(f(\tilde{\pi}(x)) = \tilde{\pi}(x)$. Consequently

$$f_W(x) = \log f'(\tilde{\pi}(x)), \quad f_H(x) = \theta \log f'(\tilde{\pi}(x)).$$

Remark 3.3. It is well known that the topological pressure of the potential $-\log f'$ is equal to 0. Properties of $\tilde{\pi}$ imply that we can view $-tf_W - f_H$ as $-(t+\theta)\log f'$. Therefore if $t + \theta = 1$ then $t = 1 - \theta$ and the box-counting dimension is equal to $2 - \theta$ by Theorem 2.

The following standard lemma establish relation between box-counting dimension of the graph of the function and the global Hölder exponent of it. Its proof can be found in [5], for example.

Lemma 3.4. *Suppose $\omega : I \to \mathbb{R}$ has the global Hölder exponent γ. Then $\gamma \leq 2 - d$, where d is the box-counting dimension of the graph of ω.*

Summarizing, Theorem 2 via Remark 3.3 and Lemma 3.4 implies the following:

Corollary 3.5. *If the strong stable manifolds of z_i do not coincide, the global Hölder exponent of function α is less or equal than θ.*

Remark 3.6. Note that this is a global result, i.e. it leaves a possibility that there is only a few points where the Hölder exponent of α can not be improved.

3.2 Proof of the upper bound for the Hölder exponent

Now we have to finish preparations for the proof of Theorem 1 that implies an "almost every" version of Corollary 3.5.

Denote for $x \in \Sigma$

$$E_n(x) = \phi_{x_0} \circ \ldots \circ \phi_{x_n}(E),$$
$$I_n(x) = f_{x_0}^{-1} \circ \ldots \circ f_{x_n}^{-1}(I).$$

For a set $A \subset Y$ denote

$$|A|_W = \sup \left\{ |x - x'| \mid (x,y),(x',y') \in A \right\},$$
$$|A|_H = \sup \left\{ |y - y'| \mid (x,y),(x',y') \in A \right\}.$$

Lemma 3.7 (Proposition 8 from [4])**.** *If the possibility 1 from Theorem 2 is not realized then there exists $N > 0$ such that for almost every $x \in \Sigma$ and $n \geq 0$*

$$|E_n(x)|_H \in [N^{-1}, N] \exp(-S_n f_H(x)).$$

Lemma 3.8. *If the possibility 1 from Theorem 2 is not realized then for μ_Σ-a.e. $x \in \Sigma$*

$$\frac{\log |E_m(x)|_H}{\log |E_m(x)|_W} \to \theta, \quad m \to \infty.$$

Proof. Note that

$$|E_m(x)|_W = \operatorname{diam} I_m(x) = a_{x_0}(x) \cdots a_{x_m}(\sigma^m(x)).$$

Then Birkhoff ergodic theorem implies that

$$\frac{\log |E_m(x)|_W}{m+1} = S_m f_W(x) \to \int f_W(x) d\mu_\Sigma(x).$$

Analogously Lemma 3.7 implies that

$$\frac{\log |E_m(x)|_H}{m+1} \to \int f_H(x) d\mu_\Sigma(x).$$

The statement now follows immediately due to the definition of functions f_W, f_H, a_i, c_i. $\square$

The following proposition together with Theorem 2 implies a version of Theorem 1 where we have $\leq$ sign instead of $=$ in the possibility 2.

Note that a similar proposition was proved in [5] (it is a part Theorem 9 there) but there b_i was asked to depend only on the first coordinate which is not true for our choice of G.

Proposition 3.9. *If the possibility 1 from Theorem 2 is not realized then for Lebesgue-almost every $x \in I$ the Hölder exponent of the function α at point x is less or equal than θ.*

Proof. Suppose $x \in I$ is such that $h_x(\alpha) > \theta$. We will show that the Lebesgue measure of such x is zero.

Choose $\theta < \gamma < \beta < \eta < h_x(\alpha)$. Then there exists a neighborhood U of x such that for every $y \in U$ $|\alpha(x) - \alpha(y)| < |x-y|^\eta < 1$. Take m large enough so that $x \in I' = I_m(\tilde{\pi}^{-1}(x)) \subset U$.

Put $a = \operatorname{argmin}_{u \in I'}$, $b = \operatorname{argmax}_{u \in I'}$. Then taking a larger m if needed

$$\begin{aligned} \left|E_m(\tilde{\pi}^{-1}(x))\right|_H = (\alpha(b) - \alpha(a)) \leq |b-a|^\eta \leq \\ \leq (\operatorname{diam} I')^\eta = \left|E_m(\tilde{\pi}^{-1}(x))\right|_W^\eta < \left|E_m(\tilde{\pi}^{-1}(x))\right|_W^\beta . \end{aligned} \tag{3.4}$$

Lemma 3.8 implies that for μ_Σ-a.e. $x \in \Sigma$ for m large enough

$$\frac{\log |E_m(x)|_H}{\log |E_m(x)|_W} \leq \gamma.$$

Therefore $|E_m(x)|_H \geq |E_m(x)|_W^\gamma$. This contradicts inequality (3.4). $\square$

Note that if b_i does not depend on the second coordinate one can prove that θ is also an upper bound (see [5]). However this proof is more technical and we will prove the upper bound in Section 4 using a simpler approach.

4 “Direct” study of Hölder exponents

Let f be a C^2-smooth endomorphism of S^1 such that $\lambda = \min_{y \in S^1} f'(y) > 1$. Let $v : S^1 \to \mathbb{R}$ be a C^1 function.

Here is the missing lower bound for the Hölder exponent. Thus this theorem completes the proof of Theorem 1.

Theorem 3. *For every positive natural number r the only bounded solution to equation* (4.7) *is θ-Hölder at every point.*

In other words $h_x(\alpha) \geq \theta$ for every $x \in S^1$.

Proof. The theorem follows from the first part of Proposition 4.7. ☐

Denote $\Lambda_1 = \max_{y \in S^1} f'(y)$.

Definition 4. We say that f is pinching with constant $\varkappa > 0$ or just pinching if $\varkappa = \Lambda_1/\lambda^2 < 1$.

The following theorem is weaker version of Theorem 1.

Theorem 4. *Suppose $v \in C^2(S^1)$ and not constant. Then there exist $0 < \varkappa_0 \leq 1$ and a natural number $r \geq 1$ such that if f is pinching with constant $\varkappa < \varkappa_0$ then the solution α to Equation* (2.1) *given by the formula* (4.7) *is not $(\theta + \gamma)$-Hölder for every $\gamma > 0$ at almost every point.*

In other words $h_x(\alpha) \leq \theta$ for almost every $x \in S^1$.

Remark 4.1. This theorem is a consequence of Proposition 3.9 but we will prove it using a completely different method.

Remark 4.2. In contrast with Remark 2.4 here if one puts $r = 1$, $f(x) = E_b(x)$ and $v(x) = \cos(2\pi x)$, then Theorem 4 implies an “almost-every” version of Hardy’s result **only if b is large enough**.

We discuss it in more detail in Remark 4.15.

Remark 4.3. Note that the pinching requirement is necessary at least for $r = 1$ because taking ϕ to be an ω-Hölder function one can use the formula for equation (2.1) to *define* v by formula (2.1): $v(x) = \alpha(\phi(x)) - (f'(x))^\theta \phi(x)$. As equation (2.1) has only one bounded solution, this implies that this solution is equal to ϕ and is therefore ω-Hölder . As ω can be taken arbitrarily it shows that the conclusion of Theorem 4 does not hold.

In other words the condition on a pinching constant is needed to ensure that possibility 2 from Theorem 1 is not realized.

4.1 Intermediate results

Theorem 4 will follow from a more general technical Proposition 4.7.

First we state a simple lemma (see [19] p169 for details) to introduce a constant that will be used later.

Lemma 4.4 (Distortion estimate)**.** *There exists a constant $C_1 \geq 1$ such that for every $x \in S^1$, every natural N and every $0 < h < 1/2$ such that $h \leq (C_1 (f^N)'(x))^{-1}$, for every $\sigma \in \{-1, 1\}$ the following estimates hold:*

$$\frac{1}{C_1} \leq \frac{(f^N)'(x)}{(f^N)'(x + \sigma h)} \leq C_1.$$

Let C_1 be a constant from Lemma 4.4. Denote

$$\Gamma_0 = \max_{y \in S^1} |v(y)|\,, \ \Gamma_1 = \max_{y \in S^1} \left|v'(y)\right|.$$

Definition 5. For a function $\phi : S^1 \to \mathbb{R}$ and $\gamma > 0$ we say that $C > 0$ is a local γ-Hölder constant for ϕ at point y if it is the infimum of $C' > 0$ such that for every $0 < h < 1$ the following estimate holds:

$$|\phi(y + h) - \phi(y)| \leq C' h^\gamma. \tag{4.1}$$

Remark 4.5. Let $r > 0$ be a positive natural number and put $\tilde{v}(x) = v(E_r(x))$. If C is a local ε-Hölder constant for v' at point y then $C r^{1+\varepsilon}$ is a local ε-Hölder constant for $\tilde{v}'(x)$.

Let $\varepsilon > 0$. Denote by Γ_2 the supremum of the local ε-Hölder constants for v' over all $x \in S^1$. It is finite if v is $C^{1+\varepsilon}$.

Let k_0 be a positive natural number.

Definition 6. Let c be a point of S^1 and v be $C^{1+\varepsilon}$. Say that the pair (f, v) satisfy condition (A) for $c \in S^1$ and $k_0 \geq 1$ if for every $0 \leq j \leq k_0 - 1$ all $v'(f^j(c))$ are either strictly positive or strictly negative simultaneously and the following estimates hold

$$\frac{6^{1+1/\varepsilon} C_1^{2-\theta} \Gamma_0 \Gamma_2^{1/\varepsilon}}{(1 - \lambda^{-k_0\theta}) \Gamma_{1,c}^{1+1/\varepsilon}} \frac{\Lambda_1^{k_0 - 1 + \theta}}{\lambda^{(k_0+1)\theta}} \leq 1; \tag{4.2}$$

$$\frac{\Gamma_1 C_1^2}{1 - \lambda^{-k_0(1-\theta)}} \frac{\Lambda_1^\theta}{\lambda^{k_0(1-\theta)+\theta}} \left(\frac{\Lambda_1}{\lambda}\right)^{j(1-\theta)} \leq$$

$$\leq v'(f^j(c))/4, \quad 0 \leq j \leq k_0 - 1; \tag{4.3}$$

$$\frac{\Gamma_{1,c}}{6\Gamma_2} \leq \Lambda_1^{k_0 - 1}, \tag{4.4}$$

where $\Gamma_{1,c} = \min_{0 \leq j \leq k_0 - 1} \left|v'(f^j(c))\right|$.

Remark 4.6. Condition (A) from Definition 6 puts rather strict restrictions on the allowed level of nonlinearity of f. If $k_0 > 1$ then morally, to satisfy all of them θ has to be close to 1, λ should be large and f should be close to linear.

If $k_0 = 1$ then condition (A) asks for the pinching constant to be small enough (see the proof of Theorem 4).

The fact that a condition on the size of λ is sufficient to get absence of Hölder continuity for some exponents in the linear case (when $f = E_\lambda$ for natural $\lambda > 1$) was mentioned in [6].

Proposition 4.7. *Let α be the only bounded solution α of equation* (2.1)*. Then*

1. *α is θ-Hölder for every natural $r \geq 1$. There is an upper bound for the local θ-Hölder constant of α at every point.*
2. *Let v be a $C^{1+\varepsilon}$ function. If $r = 1$, f is pinching and there exists a point $c \in S^1$ such that (f, v) satisfies condition (A) from Definition 6 for $c \in S^1$ and k_0 then there exists a constant $C_0 = C_0(\theta, v, f, c, k_0) > 0$ such that for almost every $x \in S^1$ for every $\hat{h}$ there exists $0 < h < \hat{h}$ such that the following lower bound holds:*

$$|\alpha(x) - \alpha(x + h)| \geq C_0 h^\theta.$$

   In particular, for almost every $y \in S^1$, the function α is not $(\theta + \gamma)$-Hölder at y for every $\gamma > 0$.

We will do the proof of this proposition for $k_0 = 1$ in Section 4.3 and for general k_0 in Appendix.

Now the proof of Theorem 4 is fairly easy.

Proof of Theorem 4. The proof readily follows from Proposition 4.7 for $k_0 = 1$ and Remark 4.5.

Put c' to be a point where the maximum of v' is attained. Put $\tilde{v} = (v \circ E_r)$. For a point $c \in E_r^{-1}(c')$ we get $\tilde{v}'(c) = rv'(c')$.

Remark 4.5 allows to choose r large enough so that the third estimate from condition (A) is satisfied. Then if the pinching constant is small enough the other two estimates from Condition (A) are satisfied as well.

Therefore (f, v) satisfy condition (A) for c and $k_0 = 1$ and Proposition 4.7 applies. □

Remark 4.8. A choice (not optimal) of constants C_0 and δ_2 can be written explicitly.

Remark 4.9. Note that the absence of Hölder continuity at almost every point does not automatically imply absence of Hölder continuity at every point since the values of h for which a lower bound as above could be written can strongly depend on a point.

Remark 4.10. Note that condition (A) does not imply pinching automatically and vice versa.

It is also possible to state a theorem very similar to part 2 of the Proposition 4.7 that guarantees lower bound for a different set of points (some of which may not belong to the set of full measure from the Proposition 4.7), with different quantifiers.

Proposition 4.11. *Let $v : S^1 \to \mathbb{R}$ be a $C^{1+\varepsilon}$ function.*

If $r = 1$, f is pinching and there exists a point c such that (f, v) satisfies condition (A) at point c for power k_0 then there exists constant $C_0 = C_0(\theta, v, f, c, k_0) > 0$ and $\delta_2 = \delta_2(\theta, v, f, c, k_0) > 0$ such that for every $\hat{h}$ there exists a natural N such that for every $x \in f^{-N}(B_{\delta_2}(c))$ there exists $0 < h < \hat{h}$ such that the following lower bound holds:

$$|\alpha(x) - \alpha(x + h)| \geq C_0 h^\theta.$$

In particular for every $\gamma > 0$ α is not $(\theta + \gamma)$-Hölder at x.

Remark 4.12. For points x that are preimages of c we can replace Γ_2 in condition (A) by the maximum of local ε-Hölder constants for v' over $0 \leq j \leq k_0 - 1$ at points $f^j(c)$.

Remark 4.13. Here is an explanation why condition (A) is needed. In the linear case $f = E_\lambda$ (this case is simpler than the general one) during the proof we will use decomposition

$$\alpha(x) - \alpha(x + h) = \sum_{j=0}^{k_0-1} B_j(x, h)$$

and prove that for every $0 < \delta_1^{(j)}, \delta_2^{(j)} < \Lambda_1^{-j}$ for some x for each j there exists positive h_j depending in an explicit way on $\delta_1^{(j)}, \delta_2^{(j)}$ such that

$$B_j(x, h_j) h_j^{-\theta} \geq K(j, \delta_1^{(j)}, \delta_2^{(j)}),$$

where $K(\cdot, \cdot, \cdot)$ is an explicit expression (see formula (6.2)). To have a *positive* lower bound for all $B_j(x, h)$, $0 \leq j \leq k_0 - 1$ for *the same* h we tune $\delta_1^{(j)}, \delta_2^{(j)}$ in such a way so that there exist δ_1, δ_2 such that

$$K(j, \delta_1, \delta_2) > 0, \quad 0 \leq j \leq k_0 - 1.$$

To be able to perform this tuning, the condition (A) is required.

Remark 4.14. It is possible to replace condition that all $v'(f^j(c))$ have the same sign by a more complicated-looking condition meaning that the sum of lower bounds for $B_j(x, h)$ for positive $v'(f^j(c))$ minus sum of upper bounds for $B_j(x, h)$ for non-positive ones is positive.

Theorem 5. *Suppose f is pinching, $v \in C^{2+\varepsilon}(S^1)$ and is not constant.*

Let c be the point where the maximum of v' is achieved. Let Γ is the local ε-Hölder constant of v'' at point c. Suppose

$$\frac{\Lambda_1^\theta C_1^2}{\lambda(1-\lambda^{\theta-1})} \leq \frac{1}{4}, \tag{4.5}$$

$$\Gamma_1 \leq 9\Gamma. \tag{4.6}$$

then for almost every x the solution α of equation (2.1) *for $r = 1$ is not $(\theta+\gamma)$-Hölder at x for every $\gamma > 0$.*

Remark 4.15. A relation to classical Weierestrass function (1) was already mentioned in Remark 4.2.

If one puts $\theta = -\log a/\log b$, $r = 1$, $f(x) = E_b(x)$, $v(x) = \cos(2\pi x)$, then $C_1 = 1$, $\varepsilon = 1$,
$\Gamma_1 = v'(1/4) = 2\pi$, $\Gamma = (2\pi)^3$ and the bounds (4.5) and (4.6) from Theorem 5 transform into

$$\frac{1}{1-b^{\theta-1}} b^{-(1-\theta)} \leq 1/4,$$

$$2\pi \leq 9(2\pi)^3.$$

Therefore Proposition 4.7 imply an "almost every"-version of Hardy's result for the case $b \geq 5^{1/(1-\theta)}$.

Now we prove a lemma stated before:

Lemma (Lemma 2.1)**.** *For every natural $r \geq 1$ there exists only one bounded solution $\alpha : S^1 \to \mathbb{R}$ to equation* (2.1)*. This solution is given by the following formula:*

$$\alpha(x) = -\sum_{i=0}^{\infty} \frac{v(E_r(f^i(x)))}{((f^{i+1})'(x))^\theta}, \quad x \in S^1. \tag{4.7}$$

Proof. Fix $x \in S^1$. It is easy to see that series (4.7) gives a bounded solution to equation (2.1).

To prove that it is the only bounded solution, suppose there is another bounded solution β. For $K = \max(\sup|\alpha|, \sup|\beta|)$ take $N \in \mathbb{N}$ so that

$$\frac{K}{((f^N)'(x))^\theta} < |\beta(x) - \alpha(x)|/3.$$

Note that for α we can write a finite analog of solution forumla (4.7):

$$\alpha(x) = \frac{-v(E_r(x))}{(f'(x))^\theta} + \frac{\alpha(f(x))}{(f'(x))^\theta} =$$
$$= \frac{-v(E_r(x))}{(f'(x))^\theta} + \frac{1}{(f'(x))^\theta}\left(\frac{-v(E_r(f(x)))}{(f'(f(x)))^\theta} + \frac{\alpha(f^2(x))}{(f'(f(x)))^\theta}\right) =$$
$$= \frac{-v(E_r(x))}{(f'(x))^\theta} - \frac{v(E_r(f(x)))}{((f^2)'(x))^\theta} + \frac{\alpha(f^2(x))}{((f^2)'(x))^\theta} = \ldots =$$
$$= -\sum_{i=0}^{N-1} \frac{v(E_r(f^i(x)))}{((f^{i+1})'(x))^\theta} + \frac{\alpha(f^N(x))}{((f^N)'(x))^\theta}.$$

Analogously

$$\beta(x) = -\sum_{i=0}^{N-1} \frac{v(E_r(f^i(x)))}{((f^{i+1})'(x))^\theta} + \frac{\beta(f^N(x))}{((f^N)'(x))^\theta}.$$

Then

$$\frac{3K}{((f^N)'(x))^\theta} < |\alpha(x) - \beta(x)| = \left|\frac{\alpha(f^N(x)) - \beta(f^N(x))}{((f^N)'(x))^\theta}\right| \leq \frac{2K}{((f^N)'(x))^\theta},$$

which is a contradiction. □

Proof of Proposition 4.7 . Let α be the only bounded solution to equation (2.1).

We show the proof first for linear f to give ideas and then for general (nonlinear) f. In each case we first prove an upper bound for $|\alpha(x) - \alpha(x+h)|\, h^{-\theta}$ for every f, then a lower bound for the case when (f, v) satisfies condition (A) for power $k_0 = 1$.

The proof of the lower bound for the case of general k_0 is put in the Appendix.

Both for linear and nonlinear case we prove several lemmas with estimates and then we use them in different combinations to study Hölder continuity properties of α.

We prove upper bounds only for the case $r = 1$ but the case of general natural $r \geq 1$ follows immediately replacing v by $v \circ E_r$.

Lemmas that will follow are meant to be inside the proof of the theorem so they inherit notations and assumptions made during the proof before they are stated.

Recall that there exists a measure with a positive density with respect to Lebesgue measure on the circle that is ergodic for f (see [19] for example). Therefore almost every point $x \in S^1$ has a dense orbit.

4.2 Proof of Proposition 4.7 in the linear case

Suppose first that $f(x) = E_\lambda(x)$, for natural $\lambda > 1$.

For every x from S^1 and $h > 0$ consider the following decomposition:

$$\alpha(x) - \alpha(x+h) = -\sum_{i=0}^{\infty} \frac{v(f^i(x))}{\lambda^{\theta(i+1)}} + \sum_{i=0}^{\infty} \frac{v(f^i(x+h))}{\lambda^{\theta(i+1)}} =$$

$$= \sum_{i=0}^{\infty} \frac{1}{\lambda^{\theta(i+1)}} \left(v(f^i(x+h)) - v(f^i(x)) \right). \tag{4.8}$$

Denote by $\mathbb{Z}_{\geq 0}$ the set of nonnegative integers and $\bar{\mathbb{Z}}_{\geq 0} = \mathbb{Z}_{\geq 0} \cup \{\infty\}$.

For $n \in \mathbb{Z}_{\geq 0}$ and $m \in \bar{\mathbb{Z}}_{\geq 0}$ such that $n \leq m$ we introduce the following notation:

$$S_{n,m}(x,h) = \sum_{i=n}^{m} \frac{1}{\lambda^{(i+1)\theta}} \left(v(f^i(x+h)) - v(f^i(x)) \right).$$

Note that for every $n \in \mathbb{Z}_{\geq 0}$ there exist $\xi_i \in (x, x+h)$ for $0 \leq i \leq n$ such that

$$S_{0,n}(x,h) = \frac{h}{\lambda^\theta} \sum_{i=0}^{n} v'(f^i(\xi_i)) \lambda^{i(1-\theta)}$$

Next we will write estimates of (4.8) from above and from below with different quantifiers for x and h.

4.2.1 Technical lemmas

Lemma 4.16. *Let N be a natural number, $0 < \delta_1 < 1$ and $h > 0$ be such that $\delta_1 \leq h\lambda^N$.*

Then for every $x \in S^1$ the following estimate holds:

$$|S_{N+1,\infty}(x,h)| \leq \frac{2\Gamma_0}{(1-\lambda^{-\theta})\lambda^{2\theta}\delta_1^\theta} h^\theta.$$

Proof of the lemma. We may easily estimate the tail of $S_{0,\infty}$ using the lower bound on h:

$$|S_{N+1,\infty}(x,h)| \leq 2\Gamma_0 \frac{1}{\lambda^{(N+2)\theta}} \sum_{i=0}^{\infty} \lambda^{-i\theta} \leq \frac{2\Gamma_0}{(1-\lambda^{-\theta})\lambda^{(N+2)\theta}} \leq \frac{2\Gamma_0}{(1-\lambda^{-\theta})\lambda^{2\theta}\delta_1^\theta} h^\theta.$$

□

Lemma 4.17. *Let N be a natural number, $0 < \delta_1, \delta_2 \leq 1$ and $h > 0$ be such that $\delta_1 \leq h\lambda^N \leq \delta_2$. Then for every $x \in S^1$ the following estimate holds:*

$$|S_{0,N}(x,h)| \leq \frac{2\Gamma_1}{1-\lambda^{\theta-1}} \delta_2^{1-\theta} \lambda^{-\theta} h^\theta.$$

If $x, c \in S^1$ are such that $\operatorname{dist}(f^N(x), c) \le \delta_2$*, and $v'(c) > 0$ then*

$$h^{-\theta}\frac{\lambda^\theta}{\delta_1^{1-\theta}}|S_{0,N}(x,h)| \ge v'(c) - 2\Gamma\delta_2^\varepsilon - \frac{2\Gamma_1}{(1-\lambda^{\theta-1})\lambda^{1-\theta}},$$

where Γ is a local ε-Hölder constant for v' at point $f^N(x)$.

Proof of the lemma. First estimate from the statement follows from the upper bound on h:

$$h^{-\theta}\frac{\lambda^\theta}{\delta_2^{1-\theta}}|S_{0,N}(x,h)| = h^{1-\theta}\frac{\lambda^\theta}{\delta_2^{1-\theta}}\left|\frac{1}{\lambda^\theta}\sum_{i=0}^{n} v'(f^i(\xi_i))\lambda^{i(1-\theta)}\right| \le$$

$$\le \sum_{i=0}^{N}\left|v'(f^i(\xi_i))\right|\frac{1}{\lambda^{(N-i)(1-\theta)}} \le \frac{2\Gamma_1}{1-\lambda^{\theta-1}}.$$

Note that we have for certain $\xi \in (x, \xi_N) \subset (x, x+h)$

$$\left|v'(f^N(\xi_N)) - v'(f^N(x))\right| \le \Gamma\left|f^N(\xi_N) - f^N(x)\right|^\varepsilon =$$

$$= \Gamma\left|(f^N)'(\xi)(\xi_N - x)\right|^\varepsilon = \Gamma(\lambda^N h)^\varepsilon \le \Gamma\delta_2^\varepsilon.$$

We also have that $\left|v'(f^N(x)) - v'(c)\right| \le \Gamma\delta_2^\varepsilon$.

Using the lower bound for h, estimate (4.1) and the condition on x we can write the following estimates:

$$h^{-\theta}\frac{\lambda^\theta}{\delta_1^{1-\theta}}|S_{0,N}(x,h)| = h^{1-\theta}\frac{\lambda^\theta}{\delta_1^{1-\theta}}\left|\frac{1}{\lambda^\theta}\sum_{i=0}^{n} v'(f^i(\xi_i))\lambda^{i(1-\theta)}\right| \ge$$

$$\ge \left|\sum_{i=0}^{N} v'(f^i(\xi_i))\frac{1}{\lambda^{(N-i)(1-\theta)}}\right| \ge$$

$$\ge \left|v'(f^N(\xi_N))\right| - \sum_{i=0}^{N-1}\left|v'(f^i(\xi_i))\right|\frac{1}{\lambda^{(N-i)(1-\theta)}} \ge$$

$$\ge v'(f^N(x)) - \Gamma\delta_2^\varepsilon - \frac{2\Gamma_1}{(1-\lambda^{\theta-1})\lambda^{1-\theta}} \ge v'(c) - 2\Gamma\delta_2^\varepsilon - \frac{2\Gamma_1}{(1-\lambda^{\theta-1})\lambda^{1-\theta}}.$$

□

4.2.2 Upper bound

First we prove that α is θ-Hölder .

Fix $x \in S^1$ and h such that $\lambda^{-1}/2 > h > 0$. Take a natural N (depending on h) such that $1/2 \le h\lambda^N \le 1$.

Then lemmas 4.16 and 4.17 for $\delta_1 = 1/2$ and $\delta_2 = 1$ imply that the following upper bound for the normalized absolute value of (4.8) holds:

$$\frac{|\alpha(x+h) - \alpha(x)|}{h^\theta} \le \frac{2\Gamma_1}{\lambda^{-\theta}(1-\lambda^{\theta-1})} + \frac{4\Gamma_0}{\lambda^{2\theta}(1-\lambda^{-\theta})}.$$

As this bound does not depend on h and N, it proves that α is θ-Hölder .

4.2.3 Lower bound when condition (A) holds for $k_0 = 1$

Assume that condition (A) is satisfied for a point c and $k_0 = 1$. Suppose without restricting generality that $v'(c) > 0$.

Fix $\hat{h}$.

We first give expressions for δ_1, δ_2 that depend only on f, v and k_0 and later select x, N and $h = h(N, \delta_1, \delta_2) < \hat{h}$ such that $|\alpha(x) - \alpha(x+h)| / h^\theta$ has a positive lower bound.

For every $0 < \delta_1, \delta_2 \leq 1$ for every x for which there exists a natural $N > 1$ such that $\left|f^N(x) - c\right| < \delta_2$, the lemmas above imply that for every $h = h(N)$ such that $\delta_1 \leq h\lambda^N \leq \delta_2$, the following lower bound holds

$$\frac{|\alpha(x+h) - \alpha(x)|}{h^\theta} \delta_1^{\theta-1} \lambda^\theta \geq$$
$$\geq v'(c) - 2\Gamma_2 \delta_2^\varepsilon - \frac{2\Gamma_1}{(1 - \lambda^{\theta-1})\lambda^{(1-\theta)}} - \frac{2\Gamma_0}{(1 - \lambda^{-\theta})\lambda^\theta \delta_1}. \tag{4.9}$$

Note that the right-hand side of this expression does not depend on x, N and h.

It is possible to choose δ_1, δ_2 independently of N, h and x in such a way that the expression above is always greater than zero. To guarantee that the second and the fourth summands in the right-hand side of (4.9) are both less than $v'(c)/3$ it is enough to put

$$\delta_1 = \frac{6\Gamma_0}{(1 - \lambda^{-\theta}) v'(c) \lambda^\theta}, \ \delta_2 = \left(\frac{v'(c)}{6\Gamma_2}\right)^{1/\varepsilon}.$$

Inequalities (4.2),(4.4) from condition (A) imply that $\delta_1 \leq \delta_2 \leq 1$.

Now fix a point x such that there exists a natural N such that

$$\left|f^N(x) - c\right| < \delta_2; \ \delta_2 \lambda^{-N} < \hat{h}.$$

One can take x to be a point from N'th-preimage of $B_{\delta_2}(c)$ for sufficiently large N or any point with a dense trajectory (the set of which has full Lebesgue measure). In the latter case x does not depend on $\hat{h}$.

Now inequality (4.3) from condition (A) and the definition of δ_1, δ_2 imply that we can select an $h < \hat{h}$ such that

$$\frac{\alpha(x) - \alpha(x+h)}{h^\theta} \geq C_0 > 0,$$

where

$$C_0 = \delta_1^{1-\theta} \lambda^{-\theta} \frac{v'(c)}{12} = \left(\frac{6\Gamma_0}{1 - \lambda^{-\theta}}\right)^{1-\theta} \frac{1}{12} \lambda^{-2\theta+\theta^2} (v'(c))^\theta.$$

in particular, C_0 does not depend on h.

4.3 Proof of Proposition 4.7 for general expanding f

Now consider a general f.

Denote $\Lambda_2 = \max_{y \in S^1} |f''(y)|$.

For every x from S^1 and $h > 0$, consider the following decomposition:

$$\alpha(x) - \alpha(x+h) = \tag{4.10}$$

$$= -\sum_{i=0}^{\infty} \frac{v(f^i(x))}{((f^{i+1})'(x))^\theta} + \sum_{i=0}^{\infty} \frac{v(f^i(x+h))}{((f^{i+1})'(x+h))^\theta} =$$

$$= \sum_{i=0}^{\infty} \frac{1}{((f^{i+1})'(x))^\theta} \left(v(f^i(x+h)) - v(f^i(x)) \right) + \tag{4.11}$$

$$+ \sum_{i=0}^{\infty} v(f^i(x+h)) \left(\frac{1}{((f^{i+1})'(x+h))^\theta} - \frac{1}{((f^{i+1})'(x))^\theta} \right). \tag{4.12}$$

4.3.1 Technical lemmas

We will need two simple formulae

Lemma 4.18. *For every $x \in S^1$ and natural $k_1 \geq k_2 \geq 0$ we have*

$$\frac{(f^{k_1})'(x)}{(f^{k_2})'(x)} = (f^{k_1-k_2})'(x)(f^{k_2}(x)). \tag{4.13}$$

Lemma 4.19. *For every $x \in S^1$ and natural $k \geq 1$ the following representation is valid:*

$$(f^k)''(x) = \sum_{p=0}^{k-1} (f^{k-p})'(x) f''\left(f^{k-p}(x)\right) \frac{(f^k)'(x)}{f'(f^{k-p}(x))} + f''(x) \frac{(f^k)'(x)}{f'(x)}. \tag{4.14}$$

Proof.

$$(f^{k+1})''(x) = \left(f(f^k(x))\right)'' = \left(f'(f^k(x))(f^k)'(x)\right)' =$$
$$= f''\left(f^k(x)\right)\left((f^k)'(x)\right)^2 + f'(f^k(x))(f^k)''(x).$$

Therefore

$$\sum_{p=0}^{k-1} \left((f^{k-p})'(x)\right)^2 f''\left(f^{k-p}(x)\right) \prod_{j=1}^{p} f'(f^{k-j}(x)) + f''(x) \prod_{j=1}^{k-1} f'(f^{k-j}(x)) =$$
$$= \sum_{p=0}^{k-1} \left((f^{k-p})'(x)\right)^2 f''\left(f^{k-p}(x)\right) \frac{(f^k)'(x)}{(f^{k-p+1})'(x)} + f''(x) \frac{(f^k)'(x)}{f'(x)}.$$

∎

Now we prove an estimate for (4.12).

Lemma 4.20. *Let $\delta_1, \delta_2 > 0$. For every $x \in S^1$ for every natural number $s > 0$ there exists $C_2(s) > 0$ such that for every natural number $N \geq s$ and $h > 0$ such that*

$$\delta_1 \leq hC_1(f^N)'(x) \leq \delta_2,$$

the following estimate holds

$$\sum_{i=0}^{\infty} v(f^i(x+h)) \left(\frac{1}{((f^{i+1})'(x+h))^\theta} - \frac{1}{((f^{i+1})'(x))^\theta} \right) \leq \tag{4.15}$$
$$\leq \frac{\left(\lambda^{-s\theta} + C_1^\theta \left(\frac{\Lambda_1}{\lambda^2}\right)^{s\theta}\right)\Gamma_0}{1-\lambda^{-\theta}} \delta_1^{-\theta} h^\theta + C_2(s)Nh.$$

Proof. Fix a natural number $s > 0$. Split (4.15) into two parts:

$$\left| \sum_{i=0}^{\infty} v(f^i(x+h)) \left(\frac{1}{((f^{i+1})'(x+h))^\theta} - \frac{1}{((f^{i+1})'(x))^\theta} \right) \right| \leq$$
$$\leq \left| \sum_{i=0}^{N+s-2} v(f^i(x+h)) \left(\frac{1}{((f^{i+1})'(x+h))^\theta} - \frac{1}{((f^{i+1})'(x))^\theta} \right) \right| + \tag{4.16}$$
$$+ \left| \sum_{i=N+s-1}^{\infty} v(f^i(x+h)) \left(\frac{1}{((f^{i+1})'(x+h))^\theta} - \frac{1}{((f^{i+1})'(x))^\theta} \right) \right|. \tag{4.17}$$

First we have to estimate (4.17) using the lower bound on h, the distortion estimate and formula (4.13).

$$\left| \sum_{i=N+s-1}^{\infty} v(f^i(x+h)) \left(\frac{1}{((f^{i+1})'(x+h))^\theta} - \frac{1}{((f^{i+1})'(x))^\theta} \right) \right| \leq$$
$$\leq \Gamma_0 \left| \sum_{i=0}^{\infty} \left(\frac{1}{((f^{N+s})'(x+h))^\theta ((f^i)'(f^{N+s}(x+h)))^\theta} - \frac{1}{((f^{N+s})'(x))^\theta ((f^i)'(f^{N+s}(x)))^\theta} \right) \right| \leq$$
$$\leq \Gamma_0 \left| \frac{1}{((f^{N+s})'(x))^\theta} \sum_{i=0}^{\infty} \left(\frac{((f^{N+s})'(x))^\theta}{((f^{N+s})'(x+h))^\theta} \frac{1}{((f^i)'(f^{N+s}(x+h)))^\theta} - \frac{1}{((f^i)'(f^{N+s}(x)))^\theta} \right) \right| \leq$$
$$\leq \Gamma_0 \frac{1 + C_1^\theta \left(\frac{\Lambda_1}{\lambda}\right)^{s\theta}}{((f^{N+s})'(x))^\theta} \sum_{i=0}^{\infty} \frac{1}{\lambda^{i\theta}} \leq \frac{\left(1 + C_1^\theta \left(\frac{\Lambda_1}{\lambda}\right)^{s\theta}\right)\Gamma_0}{\lambda^{s\theta}(1-\lambda^{-\theta})} \delta_1^{-\theta} h^\theta.$$

Now we start estimating (4.16).
Note that we have $\left|(1+h)^\theta - 1\right| \leq C_3 h$.
Since f is C^2, there exists $\xi \in (x, x+h)$ such that

$$(f^k)'(x) - (f^k)'(x+h) = -(f^k)''(\xi)\, h.$$

Using formulae (4.13), (4.14) and the distortion estimate, we can write for every natural number $1 \le k \le N$

$$\left|\frac{1}{((f^k)'(x+h))^\theta} - \frac{1}{((f^k)'(x))^\theta}\right| = \left|\frac{((f^k)'(x))^\theta - ((f^k)'(x+h))^\theta}{((f^k)'(x))^\theta((f^k)'(x+h))^\theta}\right| \le$$

$$\le \left|\frac{C_3 (f^k)''(\xi)\, h}{((f^k)'(\xi))^{1-\theta}((f^k)'(x))^\theta((f^k)'(x+h))^\theta}\right| \le$$

$$\le \frac{C_3\Lambda_2 h}{((f^k)'(\xi))^{1-\theta}((f^k)'(x))^\theta((f^k)'(x+h))^\theta}\cdot$$

$$\cdot\left(\sum_{p=0}^{k-1}(f^{k-p})'(\xi)\frac{(f^k)'(\xi)}{f'(f^{k-p}(\xi))} + \frac{(f^k)'(\xi)}{f'(\xi)}\right) =$$

$$= \frac{C_3\Lambda_2 h}{((f^k)'(x+h))^\theta}\left(\sum_{p=0}^{k-1}\frac{(f^{k-p})'(\xi)}{((f^k)'(\xi))^{1-\theta}((f^k)'(x))^\theta}\frac{(f^k)'(\xi)}{f'(f^{k-p}(\xi))} + \frac{(f^k)'(\xi)}{f'(\xi)}\right) =$$

$$= \frac{C_3\Lambda_2 h}{((f^k)'(x+h))^\theta}\left(\sum_{p=0}^{k-1}\frac{C_1^\theta}{((f^p)'(f^{k-p}(\xi)))^{1-\theta}((f^p)'(f^{k-p}(x)))^\theta}\frac{(f^k)'(\xi)}{f'(f^{k-p}(\xi))}+\right.$$

$$\left.+\frac{(f^k)'(\xi)}{f'(\xi)}\right) = \frac{((f^k)'(\xi))^\theta h}{((f^k)'(x+h))^\theta}\left(\sum_{p=0}^{k-1}\frac{C_1^\theta}{((f^p)'(f^{k-p}(\xi)))^{1-\theta}((f^p)'(f^{k-p}(x)))^\theta f'(f^{k-p}(\xi))}+\right.$$

$$\left.+\frac{1}{f'(\xi)}\right) \le C_1^\theta C_3\Lambda_2 h\left(\sum_{p=0}^{k-1}\frac{C_1^\theta}{\lambda^{p(1-\theta)+p\theta+1}} + \frac{1}{\lambda}\right) \le \frac{C_1^\theta C_3\Lambda_2}{\lambda}\left(\frac{C_1^\theta}{1-\lambda^{-1}} + 1\right) h.$$

For $k > N$ we can do a similar estimate using rough bounds instead of distortion estimates, i.e.

$$\frac{(f^{N+p})'(\xi)}{(f^{N+p})'(x)} \le C_1\left(\frac{\Lambda_1}{\lambda}\right)^p.$$

Therefore we have an estimate for (4.16) :

$$\left|\sum_{i=0}^{N+s-2} v(f^i(x+h))\left(\frac{1}{((f^{i+1})'(x+h))^\theta} - \frac{1}{((f^{i+1})'(x))^\theta}\right)\right| \le$$

$$\le \frac{C_1^\theta C_3\Lambda_2}{\lambda}\left(\frac{C_1^\theta}{1-\lambda^{-1}} + 1\right) Nh+$$

$$+\frac{C_1^\theta C_3\Lambda_2}{\lambda}C_1^\theta\left(\frac{\Lambda_1}{\lambda}\right)^{(s-2)\theta}\left(\left(\frac{\Lambda_1}{\lambda}\right)^{(s-2)\theta}\frac{C_1^\theta}{1-\lambda^{-1}} + 1\right)(s-2)h.$$

Put $C_2(s)$ to be twice the last expression divided by h. □

The rest is very similar to the linear case.

For $n \in \mathbb{Z}_{\geq 0}$ and $m \in \bar{\mathbb{Z}}_{\geq 0}$ such that $n \leq m$ we introduce the following notation for (4.11):

$$S_{n,m}(x,h) = \sum_{i=n}^{m} \frac{1}{((f^{i+1})'(x))^{\theta}} \left(v(f^i(x+h)) - v(f^i(x))\right).$$

Note that for every $n \in \mathbb{Z}_{\geq 0}$ there exist $\xi_i \in (x, x+h)$ for $0 \leq i \leq n$ such that

$$S_{0,N}(x,h) = h \sum_{i=0}^{N} v'(f^i(\xi_i)) \frac{(f^i)'(\xi_i)}{((f^{i+1})'(x))^{\theta}} =$$
$$= h \sum_{i=0}^{N} v'(f^i(\xi_i)) \frac{(f^i)'(\xi_i)}{((f^i)'(x))^{\theta}} \frac{1}{(f'(f^i(x)))^{\theta}}.$$

Lemma 4.21. *Let $x \in S^1$, let N be a natural number, $0 < \delta_1, \delta_2 < 1$ and $h > 0$ be such that $\delta_1 \leq hC_1(f^N)'(x)$. Then the following estimate holds:*

$$|S_{N+1,\infty}(x,h)| \leq \frac{2\Gamma_0}{\lambda^{2\theta}(1-\lambda^{-\theta})} \delta_1^{-\theta} h^{\theta}.$$

Proof of the lemma. We may easily estimate the tail of $S_{0,\infty}(x,h)$ using the lower bound on h:

$$|S_{N+1,\infty}(x,h)| \leq 2\Gamma_0 \frac{1}{((f^{N+2})'(x))^{\theta}} \sum_{i=0}^{\infty} \lambda^{-i\theta} =$$
$$= \frac{2\Gamma_0}{1-\lambda^{-\theta}} \frac{1}{((f^2)'(f^N(x)))^{\theta}((f^N)'(x))^{\theta}} \leq$$
$$\leq \frac{2\Gamma_0}{\lambda^{2\theta}(1-\lambda^{-\theta})} \delta_1^{-\theta} h^{\theta}.$$

□

Lemma 4.22. *Let $x \in S^1$, let N be a natural number, $0 < \delta_1, \delta_2 \leq 1$ and $h > 0$ be such that $\delta_1 \leq hC_1(f^N)'(x) \leq \delta_2$. Then*

$$|S_{0,N}(x,h)| \leq \delta_1^{1-\theta} \frac{2C_1^{\theta}\Gamma_1}{\lambda^{\theta}(1-\lambda^{\theta-1})} h^{\theta}.$$

If $x, c \in S^1$ such that $\operatorname{dist}(f^N(x), c) \leq \delta_2$, and $v'(c) > 0$ then

$$h^{-\theta} \frac{\Lambda_1^{\theta} C_1^{2-\theta}}{\delta_1^{1-\theta}} |S_N| \geq v'(c) - 2\Gamma\delta_2^{\varepsilon} - \frac{\Gamma_1 \Lambda_1^{\theta} C_1^2}{\lambda(1-\lambda^{\theta-1})},$$

where Γ is a local ε-Hölder constant for v' at point $f^N(x)$.

Proof of the lemma. First estimate from the statement follows from the upper bound on h:

$$h^{-\theta}\left(\frac{\delta_1}{C_1}\right)^{\theta-1}|S_{0,N}(x,h)| = h^{1-\theta}\left(\frac{\delta_1}{C_1}\right)^{\theta-1}\left|\sum_{i=0}^{N} v'(f^i(\xi_i))\frac{(f^i)'(\xi_i)}{((f^i)'(x))^\theta}\frac{1}{(f'(f^i(x)))^\theta}\right| \leq$$
$$\leq C_1^\theta\lambda^{-\theta}\left|\sum_{i=0}^{N} v'(f^i(\xi_i))\frac{((f^i)'(\xi_i))^{1-\theta}}{((f^N)'(x))^{1-\theta}}\right| \leq C_1\lambda^{-\theta}\left|\sum_{i=0}^{N} v'(f^i(\xi_i))\frac{1}{((f^{N-i})'(x))^{1-\theta}}\right| \leq$$
$$\leq \frac{2C_1\Gamma_1}{\lambda^\theta(1-\lambda^{\theta-1})}.$$

Note that we have for certain $\xi \in (x,\xi_N) \subset (x,x+h)$

$$\left|v'(f^N(\xi_N)) - v'(f^N(x))\right| \leq \Gamma\left|f^N(\xi_N) - f^N(x)\right|^\varepsilon =$$
$$= \Gamma\left|(f^N)'(\xi)(\xi_N - x)\right|^\varepsilon = \Gamma\left|C_1(f^N)'(x)(\xi_N - x)\right|^\varepsilon \leq \Gamma\delta_2^\varepsilon.$$

We also have $\left|v'(f^N(x)) - v'(c)\right| \leq \Gamma\delta_2^\varepsilon$. Using the distortion estimate we get

$$h^{-1}|S_{0,N}(x,h)| = \left|\sum_{i=0}^{N} v'(f^i(\xi_i))\frac{(f^i)'(\xi_i)}{((f^i)'(x))^\theta}\frac{1}{(f'(f^i(x)))^\theta}\right| \geq$$
$$\geq \left|v'(f^N(\xi_N))\right|\frac{(f^N)'(\xi_N)}{((f^N)'(x))^\theta}\frac{1}{(f'(f^N(x)))^\theta} - \sum_{i=0}^{N-1}\left|v'(f^i(\xi_i))\right|\frac{(f^i)'(\xi_i)}{((f^i)'(x))^\theta}\frac{1}{(f'(f^i(x)))^\theta} \geq$$
$$\geq \left|v'(f^N(\xi_N))\right|\frac{((f^N)'(\xi_N))^{1-\theta}}{C_1^\theta\Lambda_1^\theta} - \sum_{i=0}^{N-1}\left|v'(f^i(\xi_i))\right|\frac{C_1^\theta((f^i)'(\xi_i))^{1-\theta}}{\lambda^\theta}.$$

Using (4.13) and the lower bound on h and (4.1) we can write the following estimates:

$$h^{-\theta}|S_{0,N}(x,h)| \geq \left|v'(f^N(\xi_N))\right|\frac{\delta_1^{1-\theta}}{C_1^{2(1-\theta)+\theta}\Lambda_1^\theta} - \lambda^{-\theta}\sum_{i=0}^{N-1}\left|v'(f^i(\xi_i))\right|\frac{C_1^\theta\delta_1^{1-\theta}}{((f^{N-i})'(f^i(\xi_i)))^{1-\theta}} \geq$$
$$\geq \left|v'(f^N(x))\right|\frac{\delta_1^{1-\theta}}{C_1^{2(1-\theta)+\theta}\Lambda_1^\theta} - \Gamma\frac{\delta_1^{1-\theta}\delta_2^\varepsilon}{C_1^{2(1-\theta)+\theta}\Lambda_1^\theta} - \frac{C_1^\theta\Gamma_1}{\lambda^\theta}\sum_{i=0}^{N-1}\frac{\delta_1^{1-\theta}}{((f^{N-i})'(f^i(\xi_i)))^{1-\theta}} \geq$$
$$\geq v'(c)\frac{\delta_1^{1-\theta}}{C_1^{2-\theta}\Lambda_1^\theta} - \Gamma\frac{\delta_1^{1-\theta}\delta_2^\varepsilon}{C_1^{2-\theta}\Lambda_1^\theta} - \Gamma\frac{\delta_1^{1-\theta}\delta^\varepsilon}{C_1^{2-\theta}\Lambda_1^\theta} - C_1^\theta\delta_1^{1-\theta}\frac{\Gamma_1}{\lambda(1-\lambda^{\theta-1})}.$$

∎

4.3.2 Upper bound

First we prove that α is θ-Hölder . For every x for every $\lambda^{-1}/(2C_1) > h > 0$ take a natural number N such that $1/2 \leq hC_1(f^{N+1})'(x) \leq 1$. Then lemmas

4.20, 4.21 and 4.22 for $\delta_1 = 1/2$ and $\delta_2 = 1$ imply that the following upper bound for the normalized absolute value of (4.10) holds:

$$|\alpha(x+h) - \alpha(x)| h^{-\theta} \le C + C' N h^{1-\theta}.$$

where $C, C' > 0$ do not depend on h and N.

As this bound does not depend on h and N (the last term is negligible because h is exponentially small in N), it proves that α is θ-Hölder .

4.3.3 Lower bound when condition (A) holds for $k_0 = 1$

Assume that condition (A) is satisfied for a point c and $k_0 = 1$. Suppose without restricting generality that $v'(c) > 0$.

Fix $\hat{h}$. We first give expressions for δ_1, δ_2 that depend only on f and v and later select x, N and $h = h(N, \delta_1, \delta_2)$ such that $|\alpha(x) - \alpha(x+h)| / h^\theta$ has a positive lower bound.

For every $0 < \delta_1, \delta_2 \le 1$ for every x for which there exists a natural number $N > 1$ such that $\left|f^N(x) - c\right| < \delta_2$, the lemmas 4.20, 4.21 and 4.22 imply that for every $h = h(N)$ such that $\delta_1 \le h C_1 (f^N)'(x) \le \delta_2$, the following lower bound for the normalized absolute value of (4.10) holds:

$$\begin{aligned} \frac{|\alpha(x+h) - \alpha(x)|}{h^\theta} \frac{\Lambda_1^\theta C_1^{2-\theta}}{\delta_1^{1-\theta}} &\ge v'(c) - 2\Gamma \delta_2^\varepsilon - \frac{\Gamma_1 \Lambda_1^\theta C_1^2}{\lambda(1-\lambda^{\theta-1})} - \\ &- \left(\frac{\Lambda_1}{\lambda^2}\right)^\theta \frac{2 C_1^{2-\theta} \Gamma_0}{1-\lambda^{-\theta}} \delta_1^{-1} - \\ -\frac{\delta_1^{1-\theta}}{\Lambda_1^\theta C_1} &\left(\frac{\left(\lambda^{-s\theta} + C_1^\theta \left(\frac{\Lambda_1}{\lambda^2}\right)^{s\theta}\right) \Gamma_0}{1-\lambda^{-\theta}} \delta_1^{-\theta} h^\theta + C_2(s) N h^{1-\theta} \right). \end{aligned} \tag{4.18}$$

Put

$$\delta_1 = \left(\frac{\Lambda_1}{\lambda^2}\right)^\theta \frac{6 \Gamma_0 C_1^{2-\theta}}{(1-\lambda^{-\theta}) v'(c)}, \quad \delta_2 = \left(\frac{v'(c)}{6\Gamma_2}\right)^{1/\varepsilon}.$$

Inequalities (4.2),(4.4) from condition (A) imply that $\delta_1 \le \delta_2 \le 1$.

Now fix a point x such that there exists a natural number N such that

$$\left|f^N(x) - c\right| < \delta_2; \quad \delta_2 < \hat{h} (f^N)'(x).$$

One can take x to be a point from N'th-preimage of $B_{\delta_2}(c)$ for sufficiently large N or any point with a dense trajectory (the set of which has full Lebesgue measure). In the latter case x does not depend on $\hat{h}$.

Now inequality (4.3) from condition (A) and the definition of δ_1, δ_2 allow us to select an $h < \hat{h}$ such that

$$\frac{|\alpha(x+h)-\alpha(x)|}{h^\theta} \frac{\Lambda_{1,c}^\theta C_1^{2-\theta}}{\delta_1^{1-\theta}} \geq$$

$$\geq \frac{v'(c)}{12} - \frac{\delta_1^{1-\theta}}{\Lambda_1^\theta C_1} \left(\frac{\left(\lambda^{-s\theta} + C_1^\theta \left(\frac{\Lambda_1}{\lambda^2}\right)^{s\theta}\right)\Gamma_0}{1-\lambda^{-\theta}} \delta_1^{-\theta} h^\theta + C_2(s) N h^{1-\theta} \right).$$

Choosing s large enough and increasing N if necessary (depending on the choice of s) we can make the last summand less than $v'(c)/24$ because of pinching condition.

□

Proof of Theorem 5. Put c to be a point where the maximum of v' is attained. This implies that $v''(c) = 0$.

A modification of the proof ot Proposition 4.7 gives the necessary result.

First we modify a proof of Lemma 4.22. Note that we have for certain $\xi \in (x, \xi_N) \subset (x, x+h)$

$$\left|f^N(\xi_N) - f^N(x)\right| = \left|(f^N)'(\xi)(\xi_N - x)\right| \leq C_1 (f^N)'(x) h \leq \delta_2$$

Let Γ is the local ε-Hölder constant of v'' at point c (morally it is a third derivative of v at c). Thus

$$\left|v'(f^N(\xi_N)) - v'(f^N(x))\right| \leq$$

$$\leq \left(v''(f^N(x)) + \Gamma \left|f^N(\xi_N) - f^N(x)\right|^\varepsilon\right) \left|f^N(\xi_N) - f^N(x)\right| \leq$$

$$\leq (v''(c) + \Gamma \left|(f^N)'(x) - c\right|^\varepsilon + \Gamma \delta_2^\varepsilon)\delta_2 \leq 2\Gamma \delta_2^{1+\varepsilon}.$$

Note also that

$$\left|v'(f^N(x)) - v'(c)\right| \leq \Gamma \delta_2^\varepsilon \left|f^N(x) - c\right| \leq \Gamma \delta_2^{1+\varepsilon}.$$

Then the last estimate from the proof of Lemma 4.22 implies the following:

$$h^{-\theta} \frac{\delta_1^{1-\theta}}{\Lambda_1^\theta C_1^{2-\theta}} |S_N| \geq v'(c) - 3\Gamma \delta_2^{1+\varepsilon} - \frac{\Gamma_1 \Lambda_1^\theta C_1^2}{\lambda(1-\lambda^{\theta-1})}.$$

Then the esitimate (4.18) changes as well:

$$\frac{|\alpha(x+h)-\alpha(x)|}{h^\theta} \frac{\Lambda_1^\theta C_1^{2-\theta}}{\delta_1^{1-\theta}} \geq v'(c) - 3\Gamma\delta_2^{1+\varepsilon} - \frac{\Gamma_1 \Lambda_1^\theta C_1^2}{\lambda(1-\lambda^{\theta-1})} -$$

$$-\left(\frac{\Lambda_1}{\lambda^2}\right)^\theta \frac{2C_1^{2-\theta}\Gamma_0}{1-\lambda^{-\theta}} \delta_1^{-1} - \frac{\delta_1^{1-\theta}}{\Lambda_1^\theta C_1} \left(\frac{\left(\lambda^{-s\theta} + C_1^\theta \left(\frac{\Lambda_1}{\lambda^2}\right)^{s\theta}\right)\Gamma_0}{1-\lambda^{-\theta}} \delta_1^{-\theta} h^\theta + C_2(s) N h^{1-\theta} \right).$$

Put

$$\delta_1 = \left(\frac{\Lambda_1}{\lambda^2}\right)^\theta \frac{6\Gamma_0 C_1^{2-\theta}}{(1-\lambda^{-\theta})v'(c)}; \quad \delta_2 = \left(\frac{v'(c)}{9\Gamma}\right)^{1/1+\varepsilon}.$$

It is easy to see that $\frac{\delta_1}{\delta_2}$ is proportional to $\Gamma_0\Gamma/(v'(c))^{1+\frac{1}{1+\varepsilon}}$. Note then if one multiplies v by a constant its Hölder exponents do not change. Therefore we can assume (multiplying v by a small enough constant) that $\delta_1 \le \delta_2$.

Condition (4.5) from the statement of the Corollary implies that $\delta_2 \le 1$. Finally condition (4.6) implies that

$$\frac{\Gamma_1 \Lambda_1^\theta C_1^2}{\lambda(1-\lambda^{\theta-1})} \le \frac{v'(c)}{4}.$$

Now we can literally repeat the arguments from Subsection 4.3.3 to conclude.

□

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6 Appendix

Here we prove the rest of the Proposition 4.7 – a lower bound for

$$|\alpha(x) - \alpha(x+h)|\, h^{-\theta}$$

for general $k_0 \ge 1$.

6.1 Proof of the lower bound for general expanding f, general k_0

For a natural number $k_0 \ge 1$ for every x from S^1 and $h > 0$ we can rewrite the formula (4.7) as

$$\alpha(x) = -\sum_{j=0}^{k_0-1}\sum_{i=0}^{\infty} \frac{v(f^{k_0 i+j}(x))}{((f^{k_0 i+j+1})'(x))^\theta}.$$

Therefore

$$\alpha(x) - \alpha(x+h) =$$

$$= \sum_{j=0}^{k_0-1} B_j(x,h) + \sum_{i=0}^{\infty} v(f^i(x+h)) \left(\frac{1}{((f^{i+1})'(x+h))^\theta} - \frac{1}{((f^{i+1})'(x))^\theta} \right).$$

where

$$B_j(x,h) = \sum_{i=0}^{\infty} \frac{v(f^{k_0 i+j}(x))}{((f^{k_0 i+j+1})'(x))^\theta} - \sum_{i=0}^{\infty} \frac{v(f^{k_0 i+j}(x+h))}{((f^{k_0 i+j+1})'(x))^\theta} =$$

$$= \sum_{i=0}^{\infty} \frac{1}{((f^{(j+k_0 i+1)})'(x))^\theta} \left(v(f^{j+k_0 i}(x+h)) - v(f^{j+k_0 i}(x)) \right).$$

For $n \in \mathbb{Z}_{\geq 0}$ and $m \in \bar{\mathbb{Z}}_{\geq 0}$ such that $n \leq m$ we introduce the following notation:

$$S_{n,m}^{(j)}(x,h) = \sum_{i=n}^{m} \frac{1}{((f^{j+k_0 i+1})'(x))^\theta} \left(v(f^{k_0 i}(x+h)) - v(f^{k_0 i}(x)) \right).$$

Note that for every $0 \leq j \leq k_0 - 1$ for every $n \in \mathbb{Z}_{\geq 0}$ there exist $\xi_i = \xi_i^{(j)} \in (x, x+h)$ for $0 \leq i \leq n$ such that

$$S_{0,N}(x,h) = h \sum_{i=0}^{N} v'(f^{j+k_0 i}(\xi_i)) \frac{(f^{j+k_0 i})'(\xi_i)}{((f^{j+k_0 i+1})'(x))^\theta} =$$

$$= h \sum_{i=0}^{N} v'(f^i(\xi_i)) \frac{(f^{j+k_0 i})'(\xi_i)}{((f^{j+k_0 i})'(x))^\theta} \frac{1}{(f'(f^{j+k_0 i}(x)))^\theta}.$$

6.1.1 Technical lemmas

Lemma 6.1. *Let $x \in S^1$, let N be a natural number, $0 \leq j \leq k_0 - 1$, $0 < \delta_1, \delta_2 \leq 1$ and $h > 0$ be such that*

$$\delta_1 \leq h C_1 (f^{k_0 N})'(x).$$

Then the following estimate holds:

$$\left| S_{N+1,\infty}^{(j)}(x,h) \right| \leq \frac{2\Gamma_0}{\lambda^{(k_0+j+1)\theta}(1-\lambda^{-k_0\theta})} \delta_1^{-\theta} h^\theta.$$

Proof of the lemma. We may easily estimate the tail of $S_{0,\infty}(x,h)$ using the lower bound on h:

$$\left| S_{N+1,\infty}^{(j)}(x,h) \right| \leq 2\Gamma_0 \frac{1}{((f^{k_0(N+1)+j+1})'(x))^\theta} \sum_{i=0}^{\infty} \lambda^{-ik_0\theta} =$$

$$= \frac{2\Gamma_0}{1-\lambda^{-k_0\theta}} \frac{1}{((f^{k_0+j+1})'(f^{k_0 N}(x)))^\theta ((f^{k_0 N})'(x))^\theta} \leq$$

$$\leq \frac{2\Gamma_0}{\lambda^{(k_0+j+1)\theta}(1-\lambda^{-k_0\theta})} \delta_1^{-\theta} h^\theta.$$

$\square$

Lemma 6.2. *Let $x \in S^1$, let N be a natural number, $0 \le j \le k_0 - 1$, $0 < \delta_1, \delta_2 \le 1$, $\delta > 0$ and $h > 0$ be such that*

$$\frac{\delta_1}{C_1 (f^{k_0 N})'(x)} \le h \le \frac{\delta_2}{C_1 (f^{k_0 N})'(x)}.$$

Suppose also that $0 \le j \le k_0 - 1$ and

$$\delta_2 < \Lambda_1^{-j}. \tag{6.1}$$

If x is a point of S^1 such that $\mathrm{dist}(f^{j+k_0 N}(x), f^j(c)) \le \delta$ and $v'(f^j(c)) > 0$ then

$$h^{-\theta} \left| S_{0,N}^{(j)}(x,h) \right| \frac{C_1^{2-\theta} \Lambda_1^{\theta}}{\delta_1^{1-\theta} \lambda^{j(1-\theta)}} \ge v'(f^j(c)) - \Gamma \delta_2^{\varepsilon} \Lambda_1^{j\varepsilon} - \Gamma \delta^{\varepsilon} -$$
$$- \frac{\Gamma_1 C_1^2}{\lambda^{k_0(1-\theta)+\theta}(1 - \lambda^{-k_0(1-\theta)})} \frac{\Lambda_1^{j(1-\theta)+\theta}}{\lambda^{j(1-\theta)}},$$

where Γ is a local ε-Hölder constant for v' at point $f^{j+k_0 N}(x)$.

Proof of the lemma. Note that using the upper bound on h we obtain for some $\xi \in (x, \xi_N) \subset (x, x+h)$

$$\left| v'(f^{j+k_0 N}(\xi_N)) - v'(f^{j+k_0 N}(x)) \right| \le \Gamma \left| f^{j+k_0 N}(\xi_N) - f^{j+k_0 N}(x) \right|^{\varepsilon} =$$
$$= \Gamma \left| (f^{j+k_0 N})'(\xi)(\xi_N - x) \right|^{\varepsilon} = \Gamma \left| (f^j)'(f^{k_0 N}(\xi))(f^{k_0 N})'(\xi)(\xi_N - x) \right|^{\varepsilon} \le \Gamma \Lambda_1^{j\varepsilon} \delta_2^{\varepsilon}.$$

We also have that $\left| v'(f^{j+k_0 N}(x)) - v'(f^j(c)) \right| \le \Gamma \delta^{\varepsilon}$.

Inequality (6.1) allows us to use the distortion estimates for all indices $0 \le i \le j + k_0 N$, therefore we get

$$h^{-1} \left| S_{0,N}^{(j)}(x,h) \right| = \left| \sum_{i=0}^{N} v'(f^{j+k_0 i}(\xi_i)) \frac{(f^{j+k_0 i})'(\xi_i)}{((f^{j+k_0 i})'(x))^{\theta}} \frac{1}{(f'(f^{j+k_0 i}(x)))^{\theta}} \right| \ge$$
$$\ge \left| v'(f^{j+k_0 N}(\xi_N)) \right| \frac{(f^{j+k_0 N})'(\xi_N)}{((f^{j+k_0 N})'(x))^{\theta}} \frac{1}{(f'(f^N(x)))^{\theta}} -$$
$$- \sum_{i=0}^{N-1} \left| v'(f^{j+k_0 i}(\xi_i)) \right| \frac{(f^{j+k_0 i})'(\xi_i)}{((f^{j+k_0 i})'(x))^{\theta}} \frac{1}{(f'(f^{j+k_0 i}(x)))^{\theta}} \ge$$
$$\ge \left| v'(f^{j+k_0 N}(\xi_N)) \right| \frac{((f^{j+k_0 N})'(\xi_N))^{1-\theta}}{C_1^{\theta} \Lambda_1^{\theta}} - \sum_{i=0}^{N-1} \left| v'(f^{j+k_0 i}(\xi_i)) \right| \frac{C_1^{\theta}((f^{j+k_0 i})'(\xi_i))^{1-\theta}}{\lambda^{\theta}}$$

Using (4.13), the lower bound on h, and (4.1), we can write the following estimates:

$$
\begin{aligned}
h^{-\theta}\left|S^{(j)}_{0,N}(x,h)\right| &\geq \delta_1^{1-\theta}\frac{v'(f^{j+k_0N}(\xi_N))}{C_1\Lambda_1^\theta}\frac{((f^{j+k_0N})'(\xi_N))^{1-\theta}}{((f^{k_0N})'(x))^{1-\theta}}-\\
&-\delta_1^{1-\theta}\sum_{i=0}^{N-1}\frac{\left|v'(f^{j+k_0i}(\xi_i))\right|}{\lambda^\theta C_1^{1-2\theta}}\frac{((f^{j+k_0i})'(\xi_i))^{1-\theta}}{((f^{k_0N})'(x))^{1-\theta}}\geq\\
&\geq \delta_1^{1-\theta}\frac{\left|v'(f^{j+k_0N}(\xi_N))\right|}{C_1^{2-\theta}\Lambda_1^\theta}((f^j)'(f^{k_0N}(\xi_N)))^{1-\theta}-\\
&-\delta_1^{1-\theta}\sum_{i=0}^{N-1}\frac{\Gamma_1C_1^\theta}{\lambda^\theta}\frac{((f^j)'(f^{k_0i}(\xi_i)))^{1-\theta}}{((f^{k_0(N-i)})'(f^{k_0i}(\xi_i)))^{1-\theta}}\geq\\
&\geq \delta_1^{1-\theta}\frac{\left|v'(f^{j+k_0N}(\xi_N))\right|}{C_1^{2-\theta}\Lambda_1^\theta}\lambda^{j(1-\theta)}-\delta_1^{1-\theta}\frac{\Gamma_1C_1^\theta}{\lambda^\theta}\sum_{i=0}^{N-1}\frac{\Lambda_1^{j(1-\theta)}}{\lambda^{k_0(N-i)(1-\theta)}}\geq\\
&\geq \frac{\delta_1^{1-\theta}}{C_1^{2-\theta}\Lambda_1^\theta}\lambda^{j(1-\theta)}\left(v'(f^j(c))-\Gamma\delta_2^\varepsilon\Lambda_1^{j\varepsilon}-\Gamma\delta^\varepsilon\right)-\frac{\Gamma_1C_1^\theta\delta_1^{1-\theta}}{\lambda^{k_0(1-\theta)+\theta}(1-\lambda^{-k_0(1-\theta)})}\Lambda_1^{j(1-\theta)}.
\end{aligned}
$$

□

Assume that condition (A) is satisfied for a point c and $k_0\geq 1$. Suppose without restricting generality that $v'(c)>0$ (this implies that $v'(f^j(c))>0$ for every $0\leq j\leq k_0-1$ by condition (A)).

Fix $\hat{h}$. We first give expressions for δ_1,δ_2 that depend only on f,v and k_0 and later select x, N and $h=h(N,\delta_1,\delta_2)$ such that $\left|\alpha(x)-\alpha(x+h)\right|/h^\theta$ has a lower bound.

For every $0<\delta_1^{(j)},\delta_2^{(j)}\leq\Lambda_1^{-j}$ for every x for which there exists a natural number $N>k_0$ such that $\left|f^{k_0N}(x)-c\right|<\delta_2^{(j)}\Lambda_1^{-k_0+1}$, lemmas above (6.1 and 6.2 for $\delta=\delta_2\Lambda_1^j$) imply that for every $h=h(N)$ such that

$$
\delta_1^{(j)}\leq hC_1(f^N)'(x)\leq\delta_2^{(j)},
$$

the following bound holds for every $0\leq j\leq k_0-1$:

$$
\begin{aligned}
h^{-\theta}B_j(x,h)\frac{C_1^{2-\theta}\Lambda_1^\theta}{(\delta_1^{(j)})^{1-\theta}\lambda^{j(1-\theta)}} &\geq v'(f^j(c))-2\Gamma_2(\delta_2^{(j)})^\varepsilon\Lambda_1^{j\varepsilon}-\\
&-\frac{\Gamma_1C_1^2}{\lambda^{k_0(1-\theta)+\theta}(1-\lambda^{-k_0(1-\theta)})}\frac{\Lambda_1^{j(1-\theta)+\theta}}{\lambda^{j(1-\theta)}}-\\
&-\frac{2C_1^{2-\theta}\Lambda_1^\theta\Gamma_0}{\lambda^{(k_0+1)\theta+j}(1-\lambda^{-k_0\theta})}(\delta_1^{(j)})^{-1}.
\end{aligned}
\tag{6.2}
$$

Note that the right-hand side of this expression does not depend on x,N or h.

It is possible to choose $\delta_1^{(j)}, \delta_2^{(j)} > 0$ in such a way that the expression above is greater than zero. To guarantee that the second and the fourth summands in the right-hand side of (6.2) are both less than $v'(f^j(c))/3$ it is enough to put

$$\delta_1^{(j)} \leq D_1^{(j)} = \frac{6\Gamma_0 C_1^{2-\theta}\Lambda_1^\theta}{(1-\lambda^{-k_0\theta})\lambda^{(k_0+1)\theta+j}v'(f^j(c))},$$
$$\delta_2^{(j)} \geq D_2^{(j)} = \left(\frac{v'(f^j(c))}{6\Gamma_2}\right)^{1/\varepsilon}\frac{1}{\Lambda_1^j}.$$

To be able to select $\delta_1 \leq \delta_2$ in such a way so that they satisfy bounds above for every j but do not depend on j, we use inequality (4.2) from condition (A). It exactly means that

$$\frac{6\Gamma_0 C_1^{2-\theta}\Lambda_1^\theta}{(1-\lambda^{-k_0\theta})\lambda^{(k_0+1)\theta}\Gamma_{1,c}} = \max_j D_1^{(j)} \leq \min_j D_2^{(j)} = \left(\frac{\Gamma_{1,c}}{6\Gamma_2}\right)^{1/\varepsilon}\frac{1}{\Lambda_1^{k_0-1}},$$

where $\Gamma_{1,c} = \min_{0\leq j\leq k_0-1} v'(f^j(c))$

Put

$$\delta_1 = \max_{0\leq j\leq k_0-1} D_1^{(j)}, \quad \delta_2 = \min_{0\leq j\leq k_0-1} D_2^{(j)}.$$

Inequality (4.4) from condition (A) implies that $\delta_2 \leq \Lambda_1^{-k_0+1}$.

Now fix a point x such that there exists a natural number N such that

$$\left|f^{k_0N}(x) - c\right| < \delta_2\Lambda_1^{-k_0+1}; \quad \frac{\delta_2}{C_1(f^N)'(x)} < \hat{h}.$$

One can take x to be a point from the k_0N'th-preimage of $B_{\delta_2}(c)$ for sufficiently large N or any point with a dense trajectory (the set of which has full Lebesgue measure). In the latter case x does not depend on $\hat{h}$.

Now inequality (4.3) from condition (A) and the definition of δ_1, δ_2 allow us to select a (single) $h < \hat{h}$ such that expression (6.2) is larger than $v'(f^j(c))/12$ for every j.

It implies that

$$\frac{|\alpha(c)-\alpha(c+h)|}{h^\theta} \geq \sum_{j=0}^{k_0-1} C_1^{\theta-2}\delta_1^{1-\theta}\frac{\lambda^{j(1-\theta)}}{\Lambda_1^\theta}\frac{v'(f^j(c))}{12} -$$
$$-\frac{\delta_1^{1-\theta}}{\Lambda_1^\theta C_1}\left(\frac{\left(\lambda^{-s\theta} + C_1^\theta\left(\frac{\Lambda_1}{\lambda^2}\right)^{s\theta}\right)\Gamma_0}{1-\lambda^{-\theta}}\delta_1^{-\theta}h^\theta + C_2(s)k_0Nh\right).$$

Then as in the case of $k_0 = 1$, by choosing s large enough and increasing N if necessary (depending on the choice of s), we can make the last summand

(coming from the nonlinearity of f) less than a half of the first sum because of pinching condition.

Finally

$$\frac{|\alpha(c)-\alpha(c+h)|}{h^\theta} \geq C_0 > 0,$$

where

$$C_0 = \frac{1}{2}\sum_{j=0}^{k_0-1} C_1^{\theta-2}\delta_1^{1-\theta}\frac{\lambda^{j(1-\theta)}}{\Lambda_1^\theta}\frac{v'(f^j(c))}{12} =$$

$$= \frac{C_1^{\theta-2}}{24\Lambda_1^\theta}\left(\frac{6\Gamma_0 C_1^{2-\theta}\Lambda_1^\theta}{(1-\lambda^{-k_0\theta})\lambda^{(k_0+1)\theta}\Gamma_{1,c}}\right)^{1-\theta}\sum_{j=0}^{k_0-1}\lambda^{j(1-\theta)}v'(f^j(c)) =$$

$$= \frac{6^{1-\theta}}{24 C_1^{\theta(1-\theta)}\Lambda_1^{\theta^2}}\left(\frac{\Gamma_0}{(1-\lambda^{-k_0\theta})\Gamma_{1,c}}\right)^{1-\theta}\cdot$$

$$\cdot\lambda^{(k_0+1)\theta(\theta-1)}\sum_{j=0}^{k_0-1}\lambda^{j(1-\theta)}v'(f^j(c)).$$

in particular, C_0 does not depend on $h, \hat{h}$.

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